Threshold policies control for predator-prey systems using a control Liapunov function approach [1]

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Abstract

The stability of predator-prey models, in the context of exploitation of renewable resources, subject to threshold policies (TP) is studied in this paper using the idea of backstepping and Control Liapunov Functions (CLF) well known in control theory, as well as the concept of virtual equilibria. TPs are defined and analysed for different types of one and two species predator-prey models. The models studied are the single species Noy-Meir herbivore-vegetation model, in a grazing management context, as well as the Rosenzweig-MacArthur two species predator-prey model, in a fishery management context. TPs are shown to be versatile and useful in managing renewable resources, being simple to design and implement, and also yielding advantages in situations of overexploitation.

Key words: Predator-prey systems, Control Liapunov Function, Sustainable yield, Variable structure system, Threshold policy, Global stability, Virtual equilibrium.

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1 Introduction

Grazing management refers to the manipulation of livestock to systematically control periods of grazing and no grazing (usually termed *deferment* or *rest*). The primary objectives are to control the effects of grazing at the individual plant level in order to protect soil watershed and improve livestock production (Heitschmidt and Stuth, 1991). In grazing management, it is possible to control the consumption of the herbivore (predator) by allowing or not allowing grazing. A mathematical model that is much used in the study of herbivore grazing was proposed by Noy-Meir (1975) and will be examined in this paper. The Noy-Meir model describes vegetation growth under the assumption that it is subject to the action of a constant herbivore population. In common with most other single species models in the literature, it has a logistic growth term, and a consumption term that models the action of the herbivore.

In the grazing management context, when a scheme such as *short duration* or *deferred rotation* is used, it means that the consumption term is being switched on (when grazing of a particular paddock is allowed) and off (when the livestock is fenced out of the paddock) (Heitschmidt and Stuth, 1991). Another possibility arises in grazing models of coral reefs which can flip between coral-dominated and algae-dominated states. It has been postulated that the interplay between herbivorous fish and algae is an important factor in determining the flipping dynamics, since removal of the fish might induce an algae-dominated state. Such a proposal, based on the Noy-Meir model, of

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herbivore fish-algae dynamics, has been made in Crépin (2002).

Similarly, *a fishing policy* refers to the management of fish populations by systematically controlling the period, intensity and type of fishing. Once again, the primary objectives are to maximize productivity, without depleting or driving the stocks to extinction. In a fishery model, where two species are modeled and fishery of the prey species is of interest, it is generally not possible to control the consumption of the prey by the predator species, referred to as *endogenous consumption*, but control may be exercised by the removal (fishing) of a certain quantity of the prey species, which we will refer to as *exogenous consumption*. A mathematical model that is much used in the study of fishery is called the Gordon–Schaefer model (Clark, 1976, 1985; Imeson et al., 2002) and its variants (Collie and Spencer, 1993) are also studied in this paper. The Gordon–Schaefer model proposes a logistic growth term for the fish. Crépin (2002) proposes in addition an endogenous predation term (corresponding to herbivorous fish prey eaten by fish predators, whose population is assumed to be constant in time, thus entering as a predation rate), and a removal rate (corresponding to the removal of herbivorous fish by an exogenous agent (man)). Thus it is often true that is possible to introduce an exogenous control into either the prey or predator dynamics.

Given the complexity of ecosystem dynamics, it is only feasible to use very simple control actions. A commonly used and implementable control is to allow removal of the predator, when its density exceeds a specified threshold level – a good example of this is in a harvesting or fishing context (Collie and Spencer, 1993; Quinn and Deriso, 2000).

This paper is concerned with the introduction of such an exogenous control into one- and two-dimensional population dynamical system models. The overall objective is to develop a systematic way of designing simple implementable controls that drive the dynamical systems to a desired globally stable equilibrium, in which a desired population level is maintained and, in the case of two population models, coexistence of predator and prey population models should result, i.e., the proposed control must avoid the extinction of the species, even under certain conditions of overexploitation of the species.

This objective is attained by using the control Liapunov function (CLF) approach from the control literature (Sontag, 1989) in order to choose the control. The objective of keeping the control as simple as possible so as to be implementable is achieved by using on-off controls that are activated when a certain threshold population density increases beyond a given level. This threshold population density may be a population density itself or derived in some simple manner from these densities. The choice and positioning of the threshold is guided by the CLF as well as the concept of real and virtual equilibria introduced in Costa et al. (2000). Finally, in the case of two population predator-prey models, the simplicity of the control is achieved by introducing the control into only one of the species dynamics, and, in this case, inspired by the method of backstepping (Sepulchre et al., 1997), a CLF is used to design the control. This combination of concepts – real and virtual equilibria, CLFs, on-off control and backstepping – to introduce a globally stable equilibrium into a nonlinear dynamical population model is novel in this context and is one of the contributions of the paper. Finally, it is shown that the type of control considered in this paper has advantages in situations where overexploitation of the populations occurs, which is important in the resource management context.

In the context of fishing management, Collie and Spencer (1993) introduced a so-called *threshold policy (TP)*, which is intermediate between the well known constant escapement and constant harvest rate policies (Quinn and Deriso, 2000). A TP is defined as follows: if abundance is below the threshold level, there is no harvest; above the threshold, a constant harvest rate is applied. The TP is also referred to as an on-off control and is a special and simple case of variable structure control in the control literature (Utkin, 1978; Filippov, 1988; Utkin, 1992; Edwards and Spurgeon, 1998).

We establish a standard notation for a TP (see Fig. 1), denoting it as the function $\phi(\tau)$ defined as follows:

$$\phi(\tau) = \begin{cases} 1 \text{ if } \tau > 0 \\ 0 \text{ if } \tau < 0, \end{cases} \tag{1}$$

where τ is the threshold that should be chosen adequately, depending on the problem to be solved. The case of τ exactly equal to zero, for which the value of ϕ is not defined in (1) is discussed further below in Definition 2.

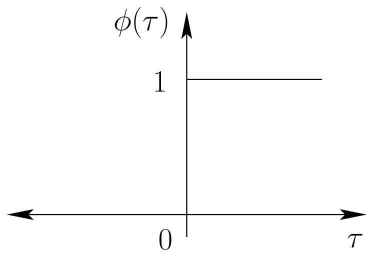


Fig. 1. Threshold policy.

One example of a TP, known as a weighted escapement policy (WEP), in which a threshold is built from a weighted (or linear) combination of prey and predator densities was proposed in Costa et al. (2000). This policy was used to stabilize a Lotka–Volterra model under simultaneous harvesting of the predator and prey. The results of this paper extend the results of Costa et al. (2000); Emel'yanov et al. (1998) by proposing a systematic design method that is applicable to all population dynamics models and, in addition, the proposed control has advantages in resource management under overexploitation.

Dynamical systems with state variables that have 'threshold' behavior (i.e., discontinuous change from one value to another when a threshold is crossed) have been studied earlier in different contexts. Below, the three main clusters of papers in each of these contexts and their differences with this paper are briefly described as follows:

(1) The main objective of Bagley and Glass (1996); Mestl et al. (1996, 1997); Edwards (2000); Gouzé and Sari (2002); Kappler et al. (2003) is the study of complex dynamics induced by existence of **endogenous** threshold switching, usually with the existence of **multiple** thresholds.
(2) In May (1977a); Matsuda et al. (1986); El-Owaidy and Moniem (2004), the term switching means a predator may direct a disproportionately large amount of its attention to the prey that happens to be most abundant at any time, i.e the predator prefers to feed itself in a habitat for some duration and changes its preference to another habitat. Once again, the emphasis is on studying complex dynamics arising from **endogenous** threshold switching, rather than designing the latter to avoid the former.
(3) In a similar vein, studies of switching in population dynamics, more specifically switching of predator behavior in the context of optimal foraging, were made in Colombo and Křivan (1993); Křivan (1996, 1997, 1998); Křivan and Sikder (1999); Boukal and Křivan (1999) and the emphasis is on the study of dynamics arising from optimal (**endogenous**)

behavior of predators.

This is in contrast with this paper, where the main emphasis is on the design of an **exogenous** control input as well as a **single** threshold chosen in such a way that complex dynamics are avoided and, moreover, a globally stable equilibrium is introduced into the dynamical system.

In control terms, in this paper, dynamical systems subject to a TP are also referred to as variable-structure systems. A TP leads to a variable-structure system with two distinct structures. In mathematical terms, this can be written as:

$$\dot{z} = f\left(z, u_\tau\right), \tag{2}$$

where z is the state vector and the control u_τ is defined as

$$u_\tau(z,t) = \begin{cases} u_1\left(z,t\right) & \text{if} \quad \tau(z) > 0 \\ \\ u_2\left(z,t\right) & \text{if} \quad \tau(z) < 0, \end{cases} \tag{3}$$

where $\tau(z)$ is a given function dependent on the state vector and a manifold M is defined as:

$$M = \left\{ z \in \mathbb{R}^2 \quad | \quad \tau(z) = 0 \right\}, \tag{4}$$

M is the set of points in where u_τ is discontinuous. Note, however, that u_1 and u_2 are continuous functions and can be interpreted as actions to remove a quantity of a determined species.

Careful mathematical consideration has to be given to the sense in which solutions to (2) are defined, given that the right-hand side may be discontinuous. The reader is referred to Edwards and Spurgeon (1998); Filippov (1988); Utkin (1992) for further details. In an ecological context, detailed discussion of discontinuous dynamics can be found in the papers of Křivan (1996); Boukal and Křivan (1999). We now define the concepts of *real* and *virtual equilibria*, for the system (2) subject to the control (3). Let G^1 and G^2 be the regions defined as $G^1 := \{z \quad | \quad \tau(z) > 0\}$, and $G^2 := \{z \quad | \quad \tau(z) < 0\}$. Also, let $z^{eq}_{G^i}$ be an equilibrium point of the dynamics in the region G^i.

Definition 1 *Let $z^{eq}_{G^i}$ be such that $f_i\left(z^{eq}_{G^i}, u_i\right) = 0$ for some u_i in (2). Then $z^{eq}_{G^i}$ is called a* **real equilibrium** *if it belongs to G^i and a* **virtual equilibrium** *if it belongs to G^j, $j \neq i$.*

From this definition, it is clear that a stable virtual equilibrium is never actually attained, since a trajectory starting, say, in G^1 and "seeking" a stable virtual equilibrium $z^{eq}_{G^1}$ located in G^2 will never attain $z^{eq}_{G^1}$ since the dynamics changes as soon as it crosses the threshold $\tau(z)$.

A **sliding mode** exists if there are regions in the vicinity of manifold M where the vectors $f_1(z, t, u_1)$ and $f_2(z, t, u_2)$ are directed toward each other (Figure 2).

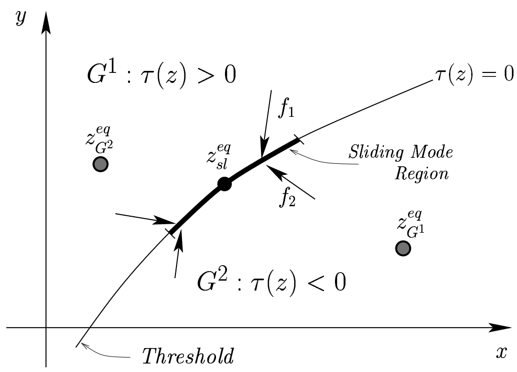


Fig. 2. Sliding modes occur on the boundary between G^1 and G^2. The sliding equilibrium z^{eq}_{sl} is shown by a bullet. Solid grey circles represent virtual stable equilibria.

The introduction of an on–off policy is responsible for new dynamic behavior, i.e. convergence to the threshold and movement along it (this is referred to as a sliding motion or sliding mode) to a point called the **sliding equilibrium**, *if it exists* (also referred to as an equilibrium attained through a **sliding mode**)

(Utkin, 1978; Filippov, 1988; Utkin, 1992).

Definition 2 *In the harvesting and grazing context, the control u_τ with a TP is defined as*

$$u_\tau = \begin{cases} u_1\,(z,t) & \textit{if} \quad \tau(z) > 0 \\ \\ 0 & \textit{if} \quad \tau(z) < 0, \end{cases}$$

where u_1 is a continuous function. The **controlled system** *is one in which the control $u_\tau = u_1$ is applied, and the* **free system** *is one in which no control (i.e., $u_\tau = 0$) is applied. Control is* **not** *defined for $\tau = 0$, and can be regarded to assume the equivalent control value described below (Utkin, 1978; Filippov, 1988; Utkin, 1992).*

Equivalent control method

Following Utkin (1992) a formal procedure will be described below to obtain equations describing sliding mode dynamics along the manifold M for the system (2).

Assume that a sliding mode exists on manifold (4). Let us find a continuous control such that, given an initial condition of the state vector on this manifold, it yields identical equality to zero of the time derivative of vectors $\tau(z)$ along system (2) trajectories:

$$\dot{\tau} = S\, f(z, u_\tau) = 0, \tag{5}$$

where $S := \{\partial\, \tau / \partial\, z\}$.

Assume that a solution of the system of algebraic equation (5) with respect to control does exist. This solution, referred to as *equivalent control* $\bar{u}_\tau(x,t)$, is substituted for u_τ in system (2)

$$\dot{z} = f(z, \bar{u}_\tau(z,t)). \tag{6}$$

Condition (5) implies that a motion starting in $\tau(z(t_0)) = 0$ will proceed along the trajectories which lie on the manifold $\tau(z) = 0$.

The above procedure will be called the *equivalent control method* and equation (6) obtained as a result of applying this method, will be regarded as the *sliding mode dynamics* describing the reduced order motion on the discontinuity surface $\tau = 0$. If the sliding mode dynamics has a stable equilibrium, then it is referred to as the sliding equilibrium or equilibrium attained through a sliding mode. Note, however, that it is not necessary that the sliding mode dynamics present a stable equilibrium (Křivan, 1996, 1998; Boukal and Křivan, 1999; Van Baalen et al., 2001).

Standard notation for the different equilibrium points of the systems will be used throughout the paper: (i) ${z_i}^{cc}$ denotes an equilibrium point of the controlled system; (ii) ${z_i}^{sc}$ denotes an equilibrium point of the free system; (iii) z_{sl}^{eq} denotes a sliding equilibrium point of the system that is reached through a sliding mode that occurs on the manifold M (Utkin, 1992).

2 Single population models with controlled exogenous consumption

In the traditional form of single population model, in which the endogenous consumption is considered together with the controlled exogenous consumption, changes in prey abundance are described by

$$
\begin{aligned}
\dot{x} &= f(x) - c_{end}(x) - c_{exo}(x)\,u, \\
&= \bar{f}(x) - c_{exo}\,(x)\,u,
\end{aligned}
\tag{7}
$$

where the continuous function $f(x)$ describes prey growth as a function of prey density, the endogenous continuous function $c_{end}(x)$ is the loss rate due to consumption either by herbivores or harvesting (the predator density is assumed constant), and the controlled exogenous consumption function is denoted by $c_{exo}(x)$, u is the control (=TP) to be designed. In other words, we choose

$$u = \varepsilon\,\phi(\tau) \tag{8}$$
$$\tau = x - x_{th} \tag{9}$$

where ε is a control effort parameter to be chosen and $\phi(\tau)$ is defined in (1) and x_{th} is the threshold value of population density.

The introduction of the term $c_{exo}\,(x)\,\varepsilon\,\phi(x - x_{th})$ means that we are choosing a control in function of prey density to be switched on and off. As far as the function $\bar{f}(x)$ in (7) is concerned, motivated by the discussion in the introduction, we consider the following forms:

$$\text{LG + No EC} \quad : \quad g\,x\left(1 - \frac{x}{x_{\max}}\right) \tag{10}$$
$$\text{LG + Holling Type II EC} \quad : \quad g\,x\left(1 - \frac{x}{x_{\max}}\right) - \frac{c_1\,x}{x + d} \tag{11}$$
$$\text{LG + Holling Type III EC} \quad : \quad g\,x\left(1 - \frac{x}{x_{\max}}\right) - \frac{c_1\,x^2}{x^2 + d^2}, \tag{12}$$

where LG means *Logistic Growth*, EC means *Endogenous Consumption*, g is the intrinsic growth rate, $x_{\max}$ is the carrying capacity, c_1 is the endogenous consumption rate, and d relates to the prey (x) density at which predator satiation occurs.

For this generalized single species model (7), we have the following theorem:

Theorem 1 *Consider a system of the type (7), with control as in (8), (9). Assume that the functions $\bar{f}(x)$ and $c_{exo}(x)$ are nonnegative on the interval $[0,\, x_m]$. Suppose that it is desired to maintain the population density x at a desired value $x_d < x_m$.*

Then, with control $\varepsilon\,\phi(\tau)$ as in (8), (9), the system stabilizes in a sliding mode equilibrium at the threshold x_{th}, so that the choice $x_{th} = x_d$ results in the desired equilibrium, if the control effort ε is chosen as follows:

$$\varepsilon > \max_{x \in [x_{th},\, x_m]} \frac{\bar{f}(x)}{c_{exo}(x)}.$$

PROOF 1 *In Appendix A.*

Remark 1 *Note that the choices of the function $\bar{f}(x)$ in (10), (11), (12) are in fact each nonnegative and bounded on an interval $[0,\, x_m]$. Models of type (7) proposed in the literature include the following: with $\bar{f}$ chosen as in (10) (Noy-Meir, 1975); as in (11) (May, 1977b; Hoekstra and Van den Bergh, 2001); as in (12) (May, 1977b; Ludwig et al., 1978; Collie and Spencer, 1993; Augustine et al., 1998).*

To appreciate and interpret this theorem, consider its application to the Noy–Meir model (i.e. $\bar{f}(x)$ as (10) logistic growth, and $c_{exo}(x) = c_2\, x$). Note that the free Noy-Meir system (i.e, without control) has the following dynamics: the origin is an unstable equilibrium point, while $x_2^{sc} = x_{\max}$ is a stable equilibrium point, and the controlled Noy–Meir system (i.e., with control) has the following dynamics: the origin is an unstable equilibrium point, and the point $x_2^{cc} =$

$(1 - c_2/g)\, x_{\max}$ is a stable equilibrium. Thus, the introduction of an on–off TP is responsible for new dynamic behavior, i.e., convergence to the threshold x_{th}, if $x_{th} > x_2^{cc}$, which is also called the **sliding equilibrium** (Utkin, 1992).

Remark 2 *It is possible to choose the threshold level x_{th}, such that $x_2^{cc} < x_{th} < x_2^{sc}$, resulting in an increase in the stabilized vegetation level. With this choice of x_{th}, the points x_2^{cc} and x_2^{sc} become virtual equilibria. In fact, it is easy to show that* ***any*** *choice of $x_{th} \in [x_2^{cc}, x_2^{sc}]$ leads to the same situation; i.e., the points x_2^{cc}, x_2^{sc} are virtual equilibria and the globally stable equilibrium under TP is x_{th}. In this sense, the choice of the threshold position is guided by studying the nature of the equilibria, i.e., all real equilibria should be unstable and any stable equilibria should be virtual, so that the only equilibrium that remains is the sliding equilibrium at x_{th}. Robustness of TPs to uncertainties in measurement can be observed in the grazing model. Such an uncertainty can occur either in the measurement of the vegetation x, and is denoted Δx, or as a small delay Δt in the switching from one value of the control ϕ to another, see Meza et al. (2002a,b,c). Essentially, any threshold position that maintains the nature of the equilibria results in stabilization of the populations.*

Remark 3 *A continuous threshold policy can also be designed in a similar manner Meza et al. (2002c), but will be omitted here for brevity.*

3 Sustainable yield for single population models with controlled exogenous consumption

This section compares sustainable yields of the model (7) that is being harvesting with a TP and without a TP. Consider a single population model (7) with $\bar{f}$ as (10) subject to grazing (or harvesting) with an exogenous consumption rate (or fishing mortality), c_2, (Clark, 1976, 1985; Kot, 2001). For this model, the concept of sustainable yield or equilibrium harvest, Y_{no-TP}, is defined as follows:

$$Y_{no-TP} = c_2\, x_2^{cc} = c_2 \left(1 - \frac{c_2}{g}\right) x_{\max}. \tag{13}$$

The graph of the logistic growth curve $gx\,(1 - x/x_{\max})$ is a concave parabola intercepting the x-axis at the origin, where it has slope g, and at the point $x_{\max}$. The consumption curve is a straight line through the origin with slope c_2. Clearly if $c_2 > g$, i.e. known as overfishing or overexploitation, then the consumption curve and the logistic curve intersect only at the **origin**, corresponding to extinction (see Fig. 3.b), which is stable. Thus, in the absence of the TP, it is necessary that the exogenous consumption rate c_2 **be less than** intrinsic growth rate g, in order that the system with constant harvest rate possess a nonzero equilibrium, which will occur at $x_2^{cc} = (1 - c_2/g)\, x_{\max}$ (see Fig. 3.a).

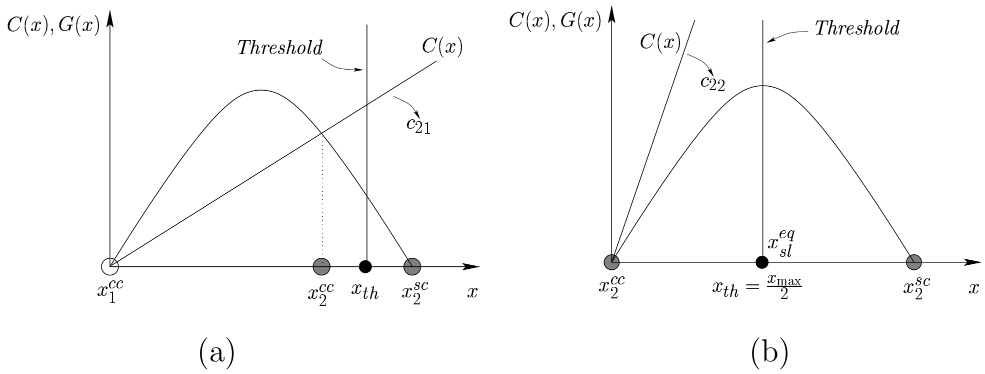


Fig. 3. (a) Equilibria with consumption curve $c(x)$ linear with medium slope ($c_{21} < g$). Free system equilibrium points – x_2^{sc}, $x_1^{sc} = 0$. Grazed system equilibrium point – x_2^{cc}, $x_1^{cc} = 0$. Parameter values: $g = 1$, $c_{21} = 0.3$, $x_{th} = 0.85$, $x_{\max} = 1$. (b) Equilibria with consumption curve $c(x)$ linear with large slope ($c_{22} > g$). Free system equilibrium point, $x_2^{sc} = x_{\max}$. Grazed system equilibrium point, $x_2^{cc} = 0$. Parameter values: $g = 1$, $c_{22} = 1.2$, $x_{th} = 0.5$, $x_{\max} = 1$.

Now, consider the same population model (7) with $\bar{f}$ as (10) and that is being harvested with a TP, $\phi(\tau)$,

$$\frac{dx}{dt} = gx\left(1 - \frac{x}{x_{\max}}\right) - \bar{c}_2\, x\, \phi(\tau). \tag{14}$$

where $\bar{c}_2$ is the **exogenous consumption rate** (or **fishing mortality**) when a TP is applied, which is defined as in (1), and $\tau(x)$ is the threshold defined as

$$\tau(x) = x - x_{th},$$

where x_{th} is the threshold level of the species, chosen as the desired equilibrium, as in theorem 1.

Since the TP is discontinuous, in order to calculate the sustainable yield of the system (14), we need to calculate the average sustainable yield, $\bar{Y}_{TP}$, at equilibrium x_{th} and this is done using the concept of **equivalent control** (Utkin, 1992), discussed in section 1, leading to the following formula:

$$\bar{Y}_{TP} = g\, x_{th}\left(1 - \frac{x_{th}}{x_{\max}}\right). \tag{15}$$

When the system (7) is subjected to a harvest with fishing mortality $c_2 > g$, **without** application of a TP this is known as overfishing or overexploitation and is a catastrophe, because, from (13), the **stock level goes to zero**, and the sustainable yield becomes

$$Y_{no-TP} = 0.$$

This shows the **advantage** of a TP (7) with $\bar{f}$ as (10) in an **overexploitation situation**. Observe, however, that the advantage of a threshold policy is only a relative one in the sense that it allows periods of overexploitation with nonzero **average** sustainable yield. Of course, the maximum average sustainable yield that results from (15) and which occurs at $x_{th} = x_{\max}/2$ is **the same** as that obtained by the application of a continuous constant harvest rate (13). However, the latter must always be below the level of overexploitation $c_2 < g$.

4 Predator-prey model subjected to an exogenous control

A large class of predator-prey models can be written as the nonlinear dynamical system

$$\begin{aligned} \dot{x} &= f_1(x) + f_2(x)\, y && (16)\\ \dot{y} &= f_3(x)\, y && (17) \end{aligned}$$

where the state variable x denotes the prey population density and the state variable y denotes the predator density; the functions f_1 and f_3 describe the prey and predator growth functions, respectively. The function f_2 describes the interaction when the predator finds the prey, i.e., the functional response. The triangular form of the system (16), (17) is known as the regular form in the control literature (Utkin, 1978, 1992).

As mentioned in the introduction, we consider the introduction of an exogenous control corresponding to the removal of the predator. The model (16), (17) therefore becomes

$$\begin{aligned} \dot{x} &= f_1(x) + f_2(x)\, y && (18)\\ \dot{y} &= f_3(x)\, y - y\, u_2 && (19) \end{aligned}$$

The remainder of this paper is devoted to showing that the proposed TP approach is successful in the control of the classical Rosenzweig–MacArthur predator-prey model that corresponds to the choice $f_1 = r\,x\,(1 - x/K)$, $f_2 = x/(x + A)$, $f_3 = s\,(x/(x + A) - J/(J + A)) = s\,A\,(x - J)/(J + A)(x + A)$ (Brauer and Soudack, 1978, 1979a,b, 1981, 1982), where r is the intrinsic growth rate of the prey, K is the carrying capacity of the environment, A is the half saturation constant, s conversion efficiency of predator, and J is the minimum prey population for which the predator can survive below the carrying capacity K, $J < K$. We choose this model because it can be regarded as one of the simplest nontrivial paradigms that was proposed after the more classical but biologically unrealistic Lotka-Volterra model. We add, however, that elsewhere we have outlined how the approach being formalized in this paper can also be used to control the Lotka-Volterra model, as well all the other predator-prey models which can be written in the form (16), (17) above (Meza et al., 2002c,b). As far as we know, most currently popular models can be written in the form (16), (17).

We state the main theorem of this section as follows.

Theorem 2 *Consider the Rosenzweig–MacArhur model*

$$\begin{cases} \dot{x}(t) &= rx\left(1 - \frac{x}{K}\right) - \frac{xy}{x+A}, \\ \dot{y}(t) &= \frac{sA(x-J)}{(J+A)(x+A)}y - y\,u_2, \\ u_2 &= \varepsilon_2\,\phi(\tau) \\ x(0) &= x_0 > 0, \quad y(0) = y_0 > 0, \end{cases} \tag{20}$$

subject to a TP defined as in (1), where τ is a threshold that has the following form

$$\tau := y - y_{th}, \tag{21}$$

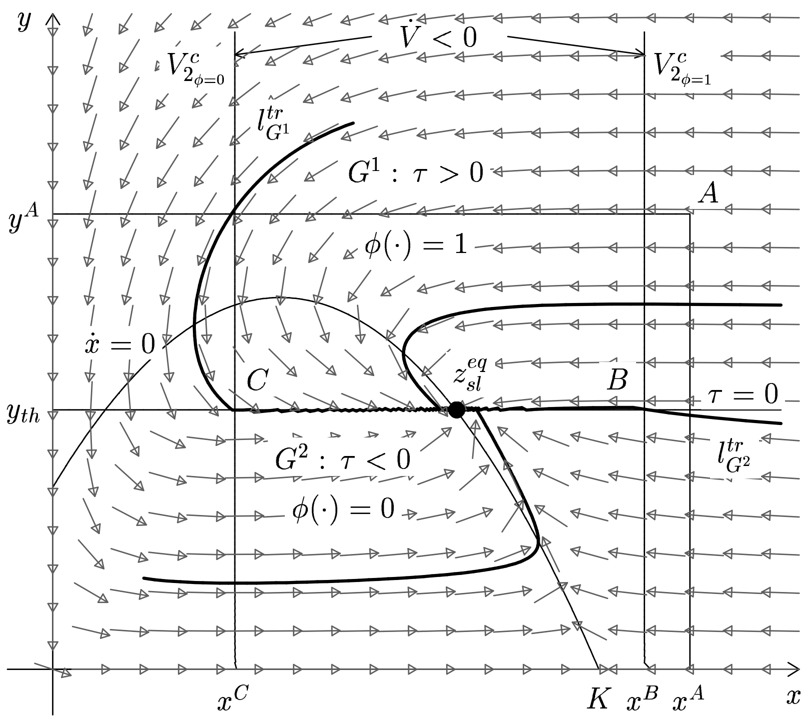


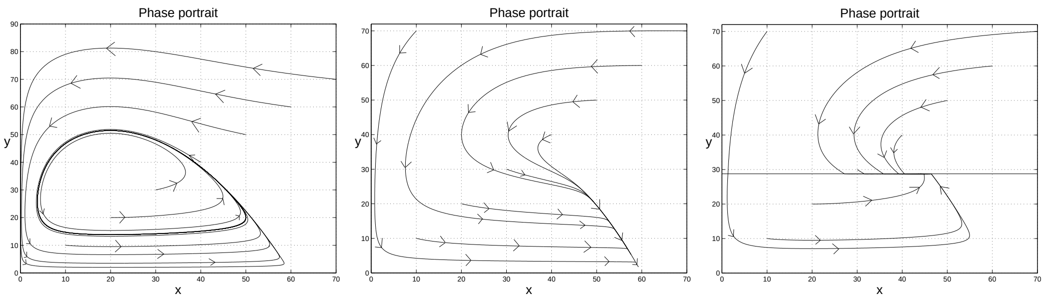
Fig. 4. The globally attracting invariant region in the phase plane is the rectangular region $0 - y^A - A - x^A - 0$. The region $G^1 := \{(x, y) : \tau > 0\}$ and $G^2 := \{(x, y) : \tau < 0\}$. The small grey arrows show the vector field. The curve labeled "$l^{tr}_{G^1}$" is the trajectory that enters the sliding domain at the point C and remains within it thenceforth, and the curve labeled "$l^{tr}_{G^2}$" is the trajectory that enters the sliding domain at the point B and remains within it thenceforth. The sliding equilibrium z^{eq}_{sl} is shown by a bullet ($\bullet$). The parameter values used in this figure are as follows: $r = 2$, $K = 60$, $A = 10$, $s = 1$, $J = 20$, $\varepsilon_2 = 0.2$, $\varepsilon = 2/5$ and $y_{th} = 28.5$. The **sliding region** is the segment CB between the curves $V^c_{\phi=0}$ and $V^c_{\phi=1}$. The prey isocline is the curve $\dot{x} = 0$.

chosen such that the prey isocline (concave parabola in Fig. 4) intersects the threshold $\tau = 0$ at a point z^{eq}_{sl} which is contained in the sliding region (segment CB in Figure 4, definition in section (B.3)). Under these conditions, z^{eq}_{sl} is a globally asymptotically stable equilibrium of (20).

PROOF 2 *Details of the proof are given in Appendix B.*

To appreciate this theorem, it should be noted that the free Rosenzweig–MacArthur system without control enters a stable limit cycle oscillation (Fig. 5.a). A conventional constant effort control leads to predator extinction (Fig.

5.b). Robust co-existence of both species (equilibrium at z_{sl}^{eq}) is, however, attained by the application of the proposed TP, systematically designed by CLFs – this is the novel observation being made in Theorem 2, which is proved in Appendix B.



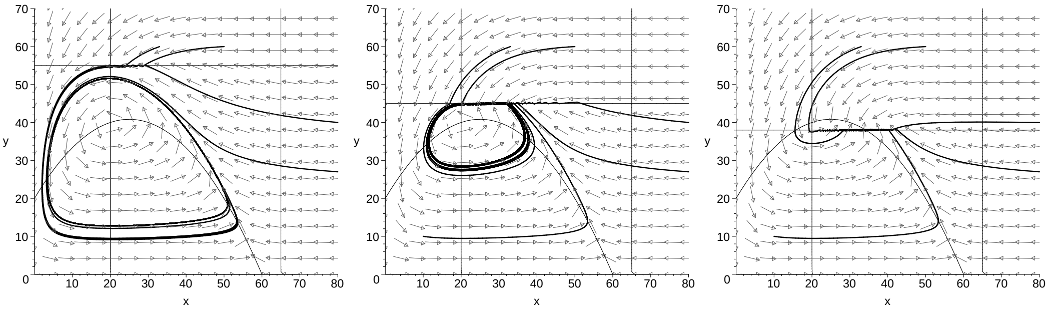
(a) (b) (c)

Fig. 5. (a) The Rosenzweig–MacArthur model **without control**. (b) Phase portrait dynamics of the Rosenzweig–MacArthur system with **proportional control** ($\varepsilon_2 = 0.2$). (c) Phase portrait dynamics of the the Rosenzweig–MacArthur system with **threshold policy** ($\varepsilon_2 = 0.2$). Parameter values: $r = 2$, $K = 60$, $s = 1$, $A = 10$ and $J = 20$.

Remark 4 *Changes in parameter values alter the isoclines and equilibrium points. However, provided the equilibria do not change their nature, the stabilization induced by TP still takes place, thus endowing this type of policy with an intrinsic robustness.*

A continuous threshold policy can also be designed in a similar manner Meza et al. (2002c), but will be omitted here for brevity.

Remark 5 *The assumption that the prey isocline intersects the threshold* $\tau = 0$ *at a point* z_{sl}^{eq} *which is contained in the sliding region corresponds to one possibility (Fig. 6.(c)). As pointed out by a reviewer, there are two other possibilities, exhibited in Figs. 6.(a) and 6.(b), in which the sliding segment of the switching line contains no sliding equilibrium. In these cases, trajectories converge to a limit cycle, part of which may be formed by the switching line itself (Fig. 6.(b)). A case similar to that shown in Fig. 6.(b) was analyzed in an* *optimal foraging context in Křivan (1996); Křivan and Sirot (1997); Křivan (1998). Demonstrations are omitted for lack of space.*



(a) (b) (c)

Fig. 6. (a) The Rosenzweig–MacArthur model (a) with $y_{th} = 55$, and z_2^{sc} is a real equilibrium. (b) with $y_{th} = 45$, and z_2^{sc} is a real equilibrium. (c) with $y_{th} = 40$, and z_2^{sc} can be virtual or real equilibrium. Parameter values are as in Figure 4.

5 Sustainable yield for the Rosenzweig-MacArthur model

Consider the model (18-19) subject to a constant harvest rate, $\varepsilon_2\, y$, on the predators as follows

$$\begin{aligned} \dot{x} &= x\left(r\left(1-\tfrac{x}{K}\right)-\tfrac{y}{x+A}\right), \\ \dot{y} &= y\left(\tfrac{sA(x-J)}{(J+A)(x+A)}-\varepsilon_2\right), \end{aligned} \tag{22}$$

where ε_2 is the control effort.

The stable equilibrium point of system (22) is:

$$\begin{aligned} x^{cc} &= \frac{A\left(\varepsilon_2(J+A)+sJ\right)}{sA-\varepsilon_2(J+A)}, \\ y^{cc} &= r\left(\frac{A\left(\varepsilon_2(J+A)+sJ\right)}{sA-\varepsilon_2(J+A)}+A\right)\left(1-\frac{A\left(\varepsilon_2(J+A)+sJ\right)}{K(sA-\varepsilon_2(J+A))}\right), \end{aligned}$$

The sustainable yield for two species models with selective harvesting at the stable equilibrium point above is studied in Beddington and May (1980);

Ströbele and Wacker (1991); Hogarth et al. (1992) and, as can be seen from Fig. 5.(b), there exist values of constant harvesting effort ε_2 such that an overfishing occurs, which leads to extinction of the predator so that the sustainable yield of the system (22) becomes zero

$$Y_{no-TP} = 0. \tag{23}$$

Now, consider the same model with harvesting of only the predator and under a TP as follows

$$\begin{aligned} \dot{x} &= x\left(r\left(1-\tfrac{x}{K}\right)-\tfrac{y}{x+A}\right), \\ \dot{y} &= y\left(\tfrac{sA(x-J)}{(J+A)(x+A)}-\bar{u}_2\right), \\ \bar{u}_2 &= \bar{\varepsilon}_2\,\phi(\tau) \end{aligned} \tag{24}$$

where $\bar{\varepsilon}_2$ is the control effort when a TP is applied to the system (24). The TP, $\phi(\tau)$, is defined as in (1), and τ is the threshold that can be defined, for example, as

$$\tau = y - y_{th}.$$

We demonstrated that the system (24) stabilizes at equilibrium $z_{sl}^{eq} = (x_{sl}^{eq},\, y_{th})$.

Since the TP is discontinuous the average sustainable yield, $\bar{Y}_{TP}$, at equilibrium z_{sl}^{eq} is calculated using the concept of equivalent control as before, yielding:

$$\bar{Y}_{TP} = y_{th}\left(\frac{sA\left(x_{sl}^{eq}-J\right)}{(J+A)\left(x_{sl}^{eq}+A\right)}\right), \tag{25}$$

which is positive for appropriate choices of the desired equilibrium (x_{sl}^{eq}, y_{th}). Once again, as in the single species case, a TP allows periods of overexploitation which maintain a nonzero average sustainable yield, which, although attainable by a constant harvest rate, would not permit overexploitation in the latter.

6 Concluding remarks

Simple on-off or threshold type policies, which are discontinuous, have been shown to be effective in the control of one species predator-prey type models commonly used in mathematical population biology. For example, in the herbivore-vegetation model under a TP, maximum herbivore consumption (and consequently, production, assuming that it is directly proportional to consumption) is guaranteed for high levels of herbivore densities, which would drive vegetation to extinction in the absence of this policy.

This paper also showed that the discontinuous threshold control called a weighted escapement policy in Costa et al. (2000) can be improved in that the control can be achieved by applying the policy to only one of the species involved. Figure 4 shows that the TP with a horizontal threshold (i.e., control applied only to the predator) stabilizes the Rosenzweig–MacArthur model at an equilibrium where there is a coexistence of species, z_{sl}^{eq}.

The sustainable yield of the system subject to a hard harvest with TP has a positive value (25), whereas, when the same system is submitted to a hard harvest with constant harvest rate, extinction of one or both species occurs, leading to a zero yield (23).

The discontinuity of the threshold policies is a drawback that could make the application of a TP somewhat impractical. We should therefore regard the proposed TP as a first step in arriving at a more realistic policy. The latter should consider different threshold for switching controls on and off for at least two reasons. First, it is likely that these thresholds are different in realistic situations. Second, delays and errors stock assessment will, in practice, result in different threshold values in practice. In control language, this means that we should consider hysteresis in the thresholds and this is currently under investigation and will be the topic of future publications.

In summary, threshold policies have been shown to be versatile and useful in managing renewable resources, being simple to design and implement, and also yielding advantages in situations of overexploitation.

Acknowledgement

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Appendixes

A Proof of the global stability theorem for the generalized one-species model

The TP, $\phi(\tau)$, with $\tau := x - x_{th}$ divides the real line $\mathbb{R}$ into two segments. In each segment the system has a different structure. Let S^1 be the segment corresponding to $\tau < 0$, where the control $u_\tau = 0$, and S^2 the segment corresponding to $\tau > 0$, where the control $u_\tau = \varepsilon$. The dynamics of the system in each segment is:

$$\text{in} \quad S^1 = [0, x_{th}] \quad \text{dynamics is} \quad \left\{ \dot{x} = \bar{f}(x), \right. \tag{A.1}$$

$$\text{in} \quad S^2 = [x_{th}, x_m] \quad \text{dynamics is} \quad \left\{ \dot{x} = \bar{f}(x) - c_{exo}(x)\,\varepsilon. \right. \tag{A.2}$$

Let the candidate control Liapunov function $V_1(x)$ be chosen as

$$V_1(x) = \frac{1}{2}(x - x_d)^2, \tag{A.3}$$

where x_d is the desired equilibrium.

Differentiating (A.3) with respect to time along the trajectories of (A.1), (A.2) yields

$$\dot{V}_1 = (x - x_d)\,\left(\bar{f}(x) - c(x)\,u\right), \tag{A.4}$$

and we do the analysis by segments:

(1) Segment S^1, $u_\tau = 0$: In this region $\dot{V}_1$ has the following form:

$$\dot{V}_1 = (x - x_d)\, \bar{f}(x). \tag{A.5}$$

Since $\bar{f}(x)$ is nonnegative, $\dot{V}_1 < 0$, $x_d = x_{th}$.

(2) Segment S^2, $u_\tau = \varepsilon$: In this region $\dot{V}_1$ has the following form,

$$\dot{V}_1 = (x - x_d)\ \left(\bar{f}(x) - c_{exo}(x)\, \varepsilon\right), \tag{A.6}$$

the control effort ε must be chosen so that the factor $\bar{f}(x) - c_{exo}(x)\, \varepsilon$ is negative, i.e.,

$$\varepsilon > \max_{x \in [x_{th},\, x_m]} \frac{\bar{f}(x)}{c_{exo}(x)}. \tag{A.7}$$

Note that, since $\bar{f}(x)/c_{exo}(x)$ is a continuous function it attains a maximum on the compact interval $[x_{th},\, x_m]$ and it is enough to choose the control effort larger than this value. The conclusion is that the system stabilizes in a sliding mode at $x = x_{th}$.

B Proof of the global stability theorem for the Rosenzweig–MacArthur model

The overall proof procedure is as follows. First, we demonstrate that the region IR $(0 - x^A - A - y^A - 0)$ is invariant and attractive. Second, we analyze the equilibria of each structure. Third, we demonstrate within the invariant region IR, all trajectories converge to sliding region of the threshold (i.e., it is also attractive), and, the latter contains a unique sliding equilibrium point, which is shown to be asymptotically stable.

B.1 Invariance of rectangular region

We will demonstrate that the rectangular region $0 - x^A - A - y^A - 0$ in Figure 4, denoted IR in the sequel, is invariant. Furthermore, this region can be chosen such that the whole "sliding domain" is within the region IR.

The following approximations

$$\begin{cases} \dot{x} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \\ \dot{y} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \end{cases} \tag{B.1}$$

are used to demonstrate the invariance of IR. The straight line $x^A A$ belongs to region G^2, with dynamics (B.3) holding and using the approximation

$$x(t+h) = x(t) + h\dot{x}.$$

A similar procedure is applied to the straight line $y^A A$. This line belongs to region G^1 with dynamics given by (B.2). The details are omitted for lack of space.

B.2 Analysis of equilibria

The threshold $\tau(z)$ divides $\mathbb{R}^2$ into two regions. In each region the system has a different structure. Let G^1 be the region corresponding to $\tau > 0$, where the control is $\varepsilon_2 y$, and G^2 the region corresponding to $\tau < 0$, where the control is 0. The dynamics of the system in each region is:

$$G^1 : \begin{cases} \dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{xy}{x+A}, \\ \dot{y} = \frac{sA(x-J)}{(J+A)(x+A)} y - \varepsilon_2 y, \end{cases} \tag{B.2}$$

$$G^2 : \begin{cases} \dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{xy}{x+A}, \\ \dot{y} = \frac{sA(x-J)}{(J+A)(x+A)} y. \end{cases} \tag{B.3}$$

The analysis of all the equilibrium points in each region is carried out. The Jacobian in region G^1 is

$$J^{G^1}_{(x,y)} = \begin{bmatrix} r - \frac{2r}{K}x - \frac{Ay}{(x+A)^2} & -\frac{x}{x+A} \\ \frac{sA(J+A)}{(J+A)(x+A)^2} y & \frac{sA(x-J)}{(J+A)(x+A)} - \varepsilon_2 \end{bmatrix}. \tag{B.4}$$

The equilibrium points of the system G^1 are as follows: $z_1^{cc} = (0,0)$ is a saddle point (with the y axis as its stable manifold and the x axis as its unstable manifold), while the point z_2^{cc} is a stable equilibrium point, which, however, does not belong to either to region G^1 or G^2 (virtual equilibrium). The point $z_3^{cc} = (K,0)$ is a saddle which belongs to region G^2.

The Jacobian in region G^2 is

$$J^{G^2}_{(x,y)} = \begin{bmatrix} r - \frac{2r}{K}x - \frac{Ay}{(x+A)^2} & -\frac{x}{x+A} \\ \frac{sA(J+A)}{(J+A)(x+A)^2} y & \frac{sA(x-J)}{(J+A)(x+A)} \end{bmatrix}. \tag{B.5}$$

The equilibrium points of the system G^2 are: $z_1^{sc} = (0,0)$, a saddle point (with the y axis as its stable manifold and the x axis as its unstable manifold), $z_2^{sc} = \big(J, \frac{r}{K}(J+A)(K-J)\big)$ is an unstable node that belongs to region G^1 (virtual equilibrium) and $z_3^{sc} = (K,0)$ is a saddle point.

Let $\tau(z) : \mathbb{R}_+^2 \to \mathbb{R}$ be a threshold, described as a function of the state vector, $z = [x \;\; y]^T$. Let the threshold τ be chosen as

$$\tau(z) = y - y_{th}. \tag{B.6}$$

The TP $\phi(\tau)$ is undefined when the state vector belongs to the set

$$M = \left\{ z \in \mathbb{R}_+^2 \;\mid\; \tau(z) = 0 \right\} \tag{B.7}$$

where M is a surface of discontinuity separating the two different structures of the system.

A sufficient condition for a sliding mode to occur on the surface of discontinuity is as follows. If a CLF $V_2(z)$ is defined as a function of τ, such that

$$V_2(z) = \frac{\tau^2(z)}{2} > 0 \;\text{ and }\; \dot{V}_2(z) = \tau \frac{\partial \tau}{\partial z} \frac{dz}{dt} < 0, \tag{B.8}$$

then a sliding mode occurs on $\tau = 0$. Let Ψ define the subset of the state space where (B.8) is satisfied:

$$\Psi = \left\{ z \in \mathbb{R}^2 \;\middle|\; \tau \frac{\partial \tau}{\partial z} \frac{dz}{dt} < 0 \right\}, \tag{B.9}$$

i.e. the domain for which $V_2(z)$ is a Liapunov function. The *sliding domain* is given by

$$\Omega = M \cap \Psi. \tag{B.10}$$

Calculating $\dot{V}_2$ we get:

$$\begin{aligned}
\dot{V}_2 &= \tau\, \dot{\tau} = \tau\, \frac{\partial \tau}{\partial z}\, \frac{dz}{dt} := \tau\, V_{2_\phi}^c, \\
&= (y - y_{th})\, [0 \quad 1] \begin{bmatrix} rx\left(1 - \frac{x}{K}\right) - \frac{xy}{x+A} \\ \frac{sA(x-J)}{(J+A)(x+A)} y - \varepsilon_2 y \phi(\cdot) \end{bmatrix} := \tau\, V_{2_\phi}^c.
\end{aligned}$$

When $\phi = 1$ then $\tau := y - \varepsilon x > 0$, therefore $V_{2_{\phi=1}}^c$ must be negative in order that $\dot{V}_2 < 0$, where

$$V_{2_{\phi=1}}^c := \frac{sA(x-J)}{(J+A)(x+A)}\, y - \varepsilon_2\, y. \tag{B.11}$$

Points that satisfy $\dot{V}_2 < 0$ are to the left of curve $V_{2_{\phi=1}}^c$. To calculate the point $B = (x_B, y_B)$, where the curve $V_{2_{\phi=1}}^c$ intersects the threshold, substitute $y = y_{th}$ in the expression for $V_{2_{\phi=1}}^c$, and solve the resulting expression for x.

When $\phi = 0$ then $\tau := y - \varepsilon x < 0$, therefore $V^c_{2_{\phi=0}}$ must be positive in order that $\dot{V} < 0$, where

$$V^c_{2_{\phi=0}} := \frac{sA(x-J)}{(J+A)(x+A)}\, y. \tag{B.12}$$

Points that satisfy $\dot{V}_2 < 0$ are to the right of this curve $V^c_{2_{\phi=0}}$. To calculate the point $C = (x_C, y_C)$, where the curve $V^c_{2_{\phi=0}}$ intersects the threshold, substitute $y = y_{th}$ in the expression for $V^c_{2_{\phi=0}}$, and solve the resulting expression for x.

Carrying out the algebraic manipulations the sliding domain turns out to be the segment CB, see Figure 4.

B.4 Sliding mode dynamics

Isocline intersects threshold in region of attractivity

This part of the proof ensures that a unique sliding equilibrium point occurs in the sliding segment, by showing that the isocline intersects the threshold in exactly one point (z^{eq}_{sl}) that lies in the region of attractivity.

The sliding equilibrium point occurs at the intersection between the threshold and the prey isocline. In the first equation of (B.2) assuming $\tau = 0$, y is substituted by y_{th} leading to:

$$rx\left(1 - \frac{x}{K}\right) - \frac{x\, y_{th}}{x+A} = 0,$$

so that the equilibrium points are

$$x^{eq}_1 = 0, \quad x^{eq}_{2,3} = \frac{r(K-A) \pm \sqrt{(r\,(K+A))^2 - 4\,r\,y_{th}\,K}}{2r}. \tag{B.13}$$

Note that $x^C < x^{eq}_{sl} < x^B$ while $x^{eq}_3 < 0 < x^C$, so that $x^{eq}_2 = x^{eq}_{sl}$ is the sliding equilibrium point.

Choice of control to ensure stability of z^{eq}_{sl}

This part of the proof shows how the control Liapunov and backstepping ideas are used to design the fictitious as well as the real input, in such as way as to guarantee global stability.

The Rosenzweig-MacArthur model with harvesting of only the predator and under a TP, $\phi(\tau)$, is shown in equation (20), where ε_2 is a control effort parameter to be chosen and $\phi(\tau)$ is defined in (1).

Hereafter, the first equation of (20) is referred to as the first subsystem and the second equation of (20) as the second subsystem.

The control Liapunov function design proceeds as follows. In the first subsystem, let $y = y_{th}$ as a fictitious input (backstepping idea). Choose the CLF such that the desired equilibrium of the first subsystem is x_d,

$$V_1 = \frac{1}{2}(x - x_d)^2$$

then

$$\dot{V}_1 = (x - x_d)\dot{x} = \frac{(x - x_d)\, x}{K(x + A)} \{r\ (K - x)\,(x + A) - K\, u_1\} = \frac{(x - x_d)\, x}{K(x + A)}\, q(x)$$

and

$$q(x) = r\ (K - x)\,(x + A) - K\, u_1$$

and, in order that $\dot{V}_1 < 0$, we must have

$$q(x) > 0, \text{ for } x < x_d \tag{B.14}$$

and

$$q(x) < 0, \text{ for } x > x_d. \tag{B.15}$$

Now assume that u_1 is proportional to the prey density x, i.e.,

$$u_1 = y_{th}. \tag{B.16}$$

Then the parameter ε needs to be chosen such that $\dot{V}_1 < 0$.

Now, u_2 needs to be chosen such that u_1 satisfies (B.16), then the equilibrium point in the first system is derived from

$$p(x) = rx\left(1 - \frac{x}{K}\right) - \frac{x\, y_{th}}{x + A} = 0.$$

Note that if (B.16) holds then

$$p(x) = \frac{x}{K(x + A)}\, q(x). \tag{B.17}$$

And the equilibrium point can be

$$x_1^{eq} = 0, \text{ or } x_{2,3}^{eq} = \frac{r(K - A) \pm \sqrt{(r\,(K + A))^2 - 4r\, y_{th}\, K}}{2r}.$$

As the desired equilibrium must have a positive value, then

$$x_d = x_2^{eq} = \frac{r(K - A) + \sqrt{(r\,(K + A))^2 - 4r\, y_t h\, K}}{2r}. \tag{B.18}$$

The final part of the proof shows that the sliding equilibrium point z_{sl}^{eq} is asymptotically stable in the sliding regime.

To demonstrate the stability of x_d we can rewrite $p(x) = 0$ as

$$p(x) = \frac{x}{K(x+A)}\, q(x) = 0$$

where

$$q(x) = -r\,x^2 + r\,(K-A)\,x + K(r\,A - y_{th}) = 0.$$

Since x_d is a zero of $q(x)$, i.e. on the left of x_d the function $q(x)$ is positive, and on the right of x_d the function $q(x)$ is negative as can be seen in Figure B.1,

$$\hat{x} = \frac{K-A}{2} < x_d.$$

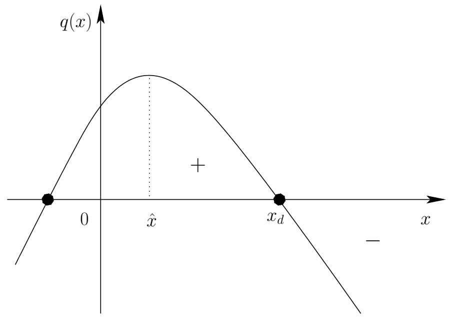

Fig. B.1. Graphical representation of $q(x)$.

Since u_1 is defined as y_{th} because it satisfies (B.14) and (B.15). Then, we demonstrate the stability of the first subsystem around the equilibrium point x_d, and in (B.13) and (B.18) we defined x_d as x_{sl}^{eq}. Therefore, we demonstrate the global stability of the system (20) around the equilibrium point z_{sl}^{eq}.

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